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ANALYSIS OF A HIERARCHICAL TWO - PERSON DIFFERENTIAL GAME*

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A hierarchical differential two-person game is formulated, with the dynamics described by a non-linear differential equation of fairly general form, and containing the terminal payoff functions. The formalization of an antagonistic differential game introduced in /1, 2/ is used to determine the optimal strategies and to reveal their structure. The optimal strategies forming the Pareto point /3/ of the set of equilibrium coalition strategies, unimprovable for the upper level player are described. The basic assumptions (declaration by the player of the upper strategy level up to the stary of the game, and the rationality of the choice of the strategy by the lower level player) go back to /4, 5/, where the static models were studied. Hierarchical dynamic games were studied, in particular, in /6—8/. The present paper is related to the work done in /1, 2, 6, 9—11/.

1. Let the controlled system be described by an equation of the form

$$x^{\cdot} = f(t, x, u, v),\ u \in P,\ v \in Q \tag{1.1}$$

where x is an n-dimensional phase vector, u and v are the vector controls of the first and second player of dimension p and q respectively, and P and Q are compacta. The function $f: G \times P \times Q \mapsto R^n$ is continuous over the set of its arguments and satifies the Lipshitz condition in x. Here G is a compactum in $R^1 \times R^n$, whose projection on the t axis is equal to the given interval $[t_0, \vartheta]$. It is assumed that all trajectories of system (1.1) originating at an arbitrary position $(t_*, x_*) \in G$ remain in G for all $t \in [t_*, \vartheta]$.

The first player strives to choose his control u at the given instant t, so that when the system is taken from its initial position $(t_*, x_*) \in G$ to the state $x[\vartheta]$ the quantity $\sigma_1(x[\vartheta])$ is minimized; at the same time the second player strives to minimize the quantity $\sigma_2(x[\vartheta])$. Here $\sigma_i: R^n \mapsto R^1$ $(i = 1, 2)$ are given continuous functions. We assume that both players know the phase vector of the system $x[t]$ at the given instant.

Below, the actions of the players in the non-antagonistic differential game will be formalized in the same classes as in the general theory of positional antagonistic differential games /1, 2/. Namely, in the general case when the saddle-point condition in a small game /1/ is not satisfied, depending on the amount of information avaialble to the players about the controls realized by the partner, we can have the following pairs of classes of player actions: {pure stategies of the first player - counterstrategies of the second player}, {mixed strategies of the first player - mixed strategies of the second player}, and {counterstrategies of the first player - pure strategies of the second player}. We shall limit our discussion, for simplicity, to the case when the condition of the saddle point in the small game holds for the function f. Then the actions of both players will be considered in the classes of pure strategies. Nevertheless, the results obtained below hold also in the general case.

We identify the strategy of the first player with the pair $U = \{u(t, x, \varepsilon),\ \beta_1(\varepsilon)\}$, where $u(t, x, \varepsilon)$ is an arbitrary function defined for $(t, x) \in G, \varepsilon > 0$, with values in P, and $\beta_1(\cdot) \in A(0, \infty)$. We denote by $A(0, \infty)$ the class of functions $\beta: (0, \infty) \mapsto (0, \infty)$ continuous and monotonic, satisfying the condition that $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The concept of the strategy as a function of position (t, x) and the accuracy parameter ε was introduced in /2/. The addition of the function $\beta_1(\cdot)$ has technical reasons, and its meaning will be discussed below at the end of Sect.2. Similarly we adopt, as the strategy of the second player, the pair $V = \{v(t, x, \varepsilon),\ \beta_2(\varepsilon)\}$ where the function $v(t, x, \varepsilon)$ is defined when $(t, x) \in G, \varepsilon > 0$ and takes values in Q, while Q, a $\beta_2(\cdot) \in A(0, \infty)$.

We further assume that the starting position $(t_*, x_*) \in G$ is fixed. Let the strategies U and V be given, and let ε_1 and ε_2 the values of the accuracy parameter ε chosen by the first and second player respectively. Let $\Delta_1 = \{\tau_i^{(1)}\}$ and $\Delta_2 = \{\tau_j^{(2)}\}$ be the partitions of the segment $[t_*, \vartheta]$ by the system of non-intersecting half-intervals $[\tau_i^{(1)}, \tau_{i+1}^{(1)})$ and $[\tau_j^{(2)}, \tau_{j+1}^{(2)})$ chosen by the first and second player respectively under the conditions that $\delta(\Delta_s) \leqslant \beta_s(\varepsilon_s)$ $(s = 1, 2)$ where $\delta(\Delta_s)$ is the step of the partition Δ_s equal to $\max_i(\tau_{i+1}^{(s)} - \tau_i^{(s)})$.

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We shall call the continuous function

$$x_\Delta{}^\varepsilon [t] = x^{\varepsilon_1, \varepsilon_2}_{\Delta_1, \Delta_2}[t, t_*, x_*, U, V]$$

satisfying the stepwise differential equation

$$\dot{x}_\Delta^\varepsilon [t] = f(t, x_\Delta{}^\varepsilon [t], u(\tau_i^{(1)}, x_\Delta{}^\varepsilon [\tau_i^{(1)}], \varepsilon_1), v(\tau_j^{(2)}, x[\tau_j^{(2)}], \varepsilon_2))$$
$$t \in [\tau_i^{(1)}, \tau_{j+1}^{(2)}) \quad (\tau_j^{(2)} \leqslant \tau_i^{(1)} < \tau_{j+1}^{(2)} \leqslant \tau_{i+1}^{(1)})$$

the Euler polygonal line generated by the strategies U and V from the starting position (t_*, x_*) for the chosen values of ε_1 and ε_2 and the partions Δ_1 and Δ_2. We shall call the continuous function $x[t] = x[t, t_*, x_*, U, V]$, for which a sequence of Euler polygonal lines $x^{\varepsilon_1^k, \varepsilon_2^k}_{\Delta_1^k, \Delta_2^k}[t, t_*{}^k, x_*{}^k, U, V]$ can be found, unformly converging to $x[t]$ on $[t_*, \vartheta]$ as $k \to \infty$, $\varepsilon_1{}^k \to 0$, $\varepsilon_2{}^k \to 0$, $t_*{}^k \to t_*$, $x_*{}^k \to x_*$, $\delta(\Delta_1{}^{(s)}) \leqslant \beta_s(\varepsilon_1{}^s)$, $s = 1, 2$, the motion generated by the strategies U and V from the starting position (t_*, x_*). Henceforth we shall assume without loss of generality, that $\varepsilon_2{}^k \leqslant \varepsilon_1{}^k$. Indeed, the chances of the second player remain undiminished, since the "scale" of its accuracy parameter can be adjusted accordingly. A pair of strategies generates, generally speaking, a bundle of motions which we shall denote by $X(t_*, x_*, U, V)$. The bundle will be a compactum in $C[t_*, \vartheta]$.

We assume that both players have complete information about the system, i.e. they know equation (1.1), the sets P and Q, and the objective functions σ_1 and σ_2.

We shall introduce the following assumptions.

A. The first player chooses his strategy $U^* = \{u^*(t, x, \varepsilon), \beta_1{}^*(\varepsilon)\}$ up to the beginning of the game, and communicates it to the second player.

B. The second player, having received the information about the strategy U^* chosen by the first player, chooses a rational strategy $V^* = \{v^*(t, x, \varepsilon), \beta_2{}^*(\varepsilon)\}$ from the condition

$$\min_V \max_{x[\cdot] \in X(t_*, x_*, U^*, V)} \sigma_2(x[\vartheta, t_*, x_*, U^*, V]) = \tag{1.2}$$
$$\max_{x[\cdot] \in X(t_*, x_*, U^*, V^*)} \sigma_2(x[\vartheta, t_*, x_*, U^*, V^*]) = \rho^*(t_*, x_*, U^*)$$

When assumptions A and B are both satisfied, we shall call the first player the upper hierarchy level player, and second the lower level player, and the differential game in question the hierarchical game.

We shall not discuss the general case of the existence of the strategy V^*. We shall merely note that rational strategies V^* exist for the strategies U^* which will be studied below. We shall denote the set of rational strategies of the second player corresponding to the disclosed strategy U^* of the first player, by $K(t_*, x_*, U^*)$. The following poperty of the rational strategies holds: for any strategy $V^* \in K(t_*, x_*, U^*)$ a motion $x_*[\cdot] \in X(t_*, x_*, U^*, V^*)$ can be found, for which

$$\sigma_2(x_*[\vartheta]) \leqslant \gamma_2(t, x_*[t]), \ \forall t \in [t_*, \vartheta] \tag{1.3}$$

Here $\gamma_2(t, x)$ is the cost function of the antagonistic game Γ_2 continuous for $(t, x) \in G$, whose dynamics are described by equation (1.1) and in which the second player dealing with the control v strives to minimize the quantity $\sigma_2(x[\vartheta])$, while the first player opposes him.

We know that such a game /2/ has universal saddle point

$$U_2{}^\circ = \{u^{(2)\circ}(t, x, \varepsilon), \beta_{12}{}^\circ(\varepsilon)\} \tag{1.4}$$
$$V_2{}^\circ = \{v^{(2)\circ}(t, x, \varepsilon), \beta_{22}{}^\circ(\varepsilon)\}$$

To formulate the condition (1.2) in approximate form, we shall make one more assumption.

C. The first player chooses at the beginning of the game the value ε_1 of his accuracy parameter and conveys it to the second player.

If assumption C holds, then condition (1.2) means that for every $\zeta > 0$ we can indicate $\varkappa(\zeta) > 0$ such that for any $\varepsilon_1 \leqslant \varkappa(\zeta)$, $\varepsilon_2 \leqslant \varepsilon_1$, and any partions Δ_1, Δ_2, $\delta(\Delta_s) \leqslant \beta_s{}^*(\varepsilon_s)$, $s = 1, 2$ the following inequality holds:

$$\sigma_2(x^{\varepsilon_1, \varepsilon_2}_{\Delta_1, \Delta_2}[\vartheta, t^*, x^*, U^*, V^*]) \leqslant \rho^*(t_*, x_*, U^*) + \zeta \tag{1.5}$$
$$((|t^* - t_*|^2 + \|x^* - x_*\|^2)^{1/2} \leqslant \min(\beta_1{}^*(\varepsilon_1), \beta_2{}^*(\varepsilon_2))$$

The fact that the second player is given the value of ε_1 only simultaneously with the start of the game, prevents him from using this information to sharpen his rational strategy. The information about ε_1 is used by the second player in choosing the value ε_2 of his own accuracy parameter so as to ensure the inequality $\varepsilon_2 \leqslant \varepsilon_1$ during the construction of the Euler polygonal lines.

Thus when the first player discloses his strategy U, the second player has at his disposal a set of rational strategies $K(t_*, x_*, U)$, and any strategy $V \in K(t_*, x_*, U)$ guarantees the second player a result equal to $\rho^*(t_*, x_*, U)$. At the same time, the result obtained by the first player depends, generally speaking, on what strategy the second player selects from the set $K(t_*, x_*, U)$.

We shall distinguish between two cases.

Case 1. The second player chooses a rational strategy from the set $K(t_*, x_*, U)$ in an arbitrary manner.

Clearly, in case 1 the guaranteed result of the first player upon his disclosure of the strategy U, will be

$$\sup_{V\in K(t_*, x_*, U)} \max_{x[\cdot]\in X(t_*, x_*, U, V)} \sigma_1(x[\vartheta, t_*, x_*, U, V]) = \rho^{(1)}(U) \tag{1.6}$$

Case 2. The second player shows goodwill towards the first player and chooses the rational strategy V from the condition

$$\min_{V_*\in K(t_*, x_*, U)} \max_{x[\cdot]\in X(t_*, x_*, U, V_*)} \sigma_1(x[\vartheta, t_*, x_*, U, V_*]) = \max_{x[\cdot]\in X(t_*, x_*, U, V)} \sigma_1(x[\vartheta, t_*, x_*, U, V]) = \rho^{(2)}(U) \tag{1.7}$$

In (1.7) we write **min** and not **inf**, in view of the fact that for the strategies U dealt with here, the minimum is attained on the set $K(t_*, x_*, U)$. The quantity $\rho^{(2)}(U)$ will represent a guaranteed result for the first player when he discloses his strategy in case 2.

We shall agree, below, to speak simply of a hierarchical differential game in case 1, and of a hierarchical differential game with a benevolent second player in case 2.

Problem 1. To find the strategy of the first player $U^\circ = \{u^\circ(t, x, \varepsilon), \beta_1^\circ(\varepsilon)\}$ such that

$$\rho^{(1)}(U^\circ) = \min\nolimits_U \rho^{(1)}(U) \tag{1.8}$$

Problem 2. To find the strategy of the first player $U_0 = \{u_0(t, x, \varepsilon), \beta_{10}(\varepsilon)\}$, such that

$$\rho^{(2)}(U_0) = \min\nolimits_U \rho^{(2)}(U) \tag{1.9}$$

Definition 1. We shall call the strategy U°, which is a solution of Problem 1, the optimal strategy of the first player in a hierarchical differential game. We shall call the otpimal strategy of the second any strategy from the set $K(t_*, x_*, U^\circ)$.

Definition 2. We shall call the strategy U_0, which is a solution of Problem 2, the optimal strategy of the first player in a hierarchical differential game with a benevolent second player. We shall call the optimal strategy of the benevolent second player, any strategy from the set $K(t_*, x_*, U_0)$ satisfying the condition (1.7).

Let us formulate the following auxiliary optimal-control problem.

Problem 3. Let the dynamics of the controlled system be described by Eq.(1.1). The problem is to find a pair of measurable functions $(u(t), v(t), t_* \leqslant t \leqslant \vartheta)$, furnishing the quantity $\sigma_1(x(\vartheta))$ with a minimum subject to the condition

$$\sigma_2(x(\vartheta)) \leqslant \gamma_2(t, x(t)), \quad t_* \leqslant t \leqslant \vartheta \tag{1.10}$$

where $\gamma_2(t, x)$ is a cost function of the antagonistic game Γ_2 and $x(t), t_* \leqslant t \leqslant \vartheta$ is a trajectory of system (1.1) generated by the controls $u(\cdot)$ and $v(\cdot)$ from the starting position (t_*, x_*). The trajectories of system (1.1) satisfying inequality (1.10) will be called admissible. It can be shown that the set of admissible trajectories is non-empty and compact in the metric of the space $C[t_*, \vartheta]$, provided that we also assume that the vectogram

$$Q^*(t, x) = \{q \in R^n : q = f(t, x, u, v), u \in P, v \in Q\} \tag{1.11}$$

is convex. Then a solution to Problem 3 exists.

2. Let a pair of measurable functions $u^*(\cdot), v^*(\cdot)$ supply the solution of Problem 3 with $x^*(\cdot)$ representing the corresponding trajectory. Using Luzin's theorem, we can find families of continuous functions $\{u^\varepsilon(\cdot)\}, \{v^\varepsilon(\cdot)\}$, such that $\|x^\varepsilon(t) - x^*(t)\| \leqslant \varepsilon$ for all $t \in [t_*, \vartheta]$. Here $x^\varepsilon(\cdot)$ is a trajectory of system (1.1) generated by the controls $u^\varepsilon(\cdot), v^\varepsilon(\cdot)$.

Let us consider the strategies of the first and second player $U^* = \{u^*(t, x, \varepsilon), \beta_1^*(\varepsilon)\}$ and $V^* = \{v^*(t, x, \varepsilon), \beta_1^*(\varepsilon)\}$, where

$$u^*(t, x, \varepsilon) = \begin{cases} u^\varepsilon(t), & \|x - x^\varepsilon(t)\| \leqslant \varepsilon \\ u^{(2)*}(t, x, \varepsilon), & \|x - x^\varepsilon(t)\| > \varepsilon \end{cases} \tag{2.1}$$

$$v^*(t, x, \varepsilon) = \begin{cases} v^\varepsilon(t), & \|x - x^\varepsilon(t)\| \leqslant \varepsilon \\ v^{(2)*}(t, x, \varepsilon), & \|x - x^\varepsilon(t)\| > \varepsilon \end{cases} \tag{2.2}$$

$$\forall\, t \in [t_*, \vartheta], \quad \varepsilon > 0$$

and the majorant $\beta_1^*(\varepsilon)$ common to both strategies is chosen so that the following inequality holds:

$$\| x_{\Delta_1,\Delta_2}^{\varepsilon_1,\varepsilon_2}[t, t_*, x_*, U^*, V^*] - x^{\varepsilon}(t) \| < \varepsilon, \quad \forall t \in [t_*, \vartheta] \tag{2.3}$$

for the Euler polygonal lines at $\varepsilon_2 \leqslant \varepsilon$, $\delta(\Delta_1) \leqslant \beta_1^*(\varepsilon)$, $\delta(\Delta_2) \leqslant \beta_1^*(\varepsilon_2)$. The functions $u^{(2)\circ}(t, x, \varepsilon)$, $v^{(2)\circ}(t, x, \varepsilon)$ are defined in (1.4).

It can be confirmed that $V^* \in K(t_*, x_*, U^*)$. We note that the bundle $X(t_*, x_*, U^*, V^*)$ consists of the unique trajectory $x^*(\cdot)$.

Let us write $\tau_* = \min\{\tau \in [t_*, \vartheta] : \sigma_2(x^*(\vartheta)) = \gamma_2(\tau, x^*(\tau))\}$. Two cases are possible.

a) $\tau_* = \vartheta$. Interestingly, this means that the second player, having obtained the information about the disclosed strategy of the first player U^*, is now interested in tracing the trajectory $x^*(\cdot)$ up to the termination of the game. We note that the problem studied in /11/ refers to precisely this case.

Theorem 1. Let the assumptions A and B both hold. Let the pair of measurable functions $u^*(\cdot)$, $v^*(\cdot)$ solve Problem 3, and let condition $\tau_* = \vartheta$ hold for the corresponding trajectory. Then the strategies U^*, V^* (2.1)–(2.3) in the hierarchical differential game are optimal.

Proof. As was already noted, $V^* \in K(t_*, x_*, U^*)$. Any other strategy from the set $K(t_*, x_*, U^*)$ will also ensure the tracing of the trajectory $x^*(\cdot)$. Therefore by disclosing U^* the first player guarantees for himself the result $\rho^{(1)}(U^*) = \sigma_1(x^*(\vartheta))$. We shall show that this result cannot be improved by the first player. Let us assume the opposite, namely, that a strategy of the first player U^+ and a number $\mu > 0$ exist such that the following inequality holds for any strategy of the second player $V^+ \in K(t_*, x_*, U^+)$, and any motion $x[\cdot] \in X(t_*, x_*, U^+, V^+)$:

$$\sigma_1(x[\vartheta]) \leqslant \rho^{(1)}(U^*) - \mu \tag{2.4}$$

In particular, (2.4) holds for the motion $x^+[\cdot] \in X(t_*, x_*, U^+, V^+)$ for which inequality (1.3) holds, i.e. $\sigma_2(x^+[\vartheta]) \leqslant \gamma_2(t, x^+[t])$ at all $t \in [t_*, \vartheta]$.

Taking into account the assumption that the vectogram $Q^*(t, x)$ (1.9) is convex, we can conclude that measurable functions $u_*(t), v_*(t), t_* \leqslant t \leqslant \vartheta$ exist, generating for the starting position (t_*, x_*) the trajectory $x_*(\cdot)$, which coincides with the trajectory $x^+[\cdot]$. From this it follows that the trajectory $x_*(\cdot)$ is admissible in Problem 3. But the fact that inequality (2.4) holds for this problem contradicts the statement that $u^*(\cdot), v^*(\cdot)$ is a solution of Problem 3.

b) $\tau_* < \vartheta$. Now we cannot assert any more that the strategies U^*, V^* (2.1)–(2.3) will be optimal in the hierarchical differential game, since the second player is interested in tracing the trajectory $x^*(\cdot)$ only up to the instant τ_*. From the instant τ_* onwards he can "switch over" to the strategy $V^{(2)*}$ (1.4) guaranteeing for himself the result $\sigma_2(x^*(\vartheta))$. Such a switch over may be found to be undesirable for the first player. If on the other hand we consider the hierarchical differential game with benevolent seoond player, then the following theorem holds.

Theorem 2. Let assumptions A and B both hold, and let the pair of measurable functions $u^*(\cdot), v^*(\cdot)$ solve Problem 3, with the inequality $\tau_* < \vartheta$ holding for the corresponding trajectory. Then the strategies U^*, V^* (2.1)–(2.3) in the hierarchical differential game with benevolent second player will be optimal.

The theorem is proved in the same manner as Theorem 1, with any changes made fully understood.

Thus we can come to the following conclusion. If amongst the solutions of Problem 3 $u^*(\cdot), v^*(\cdot)$ then Problem 1 has a solution represented by the strategy U^* (2.1), (2.3). The strategy U^* will be a solution of Problem 2 even more so. If on the other hand we have the inequality $\tau_* < \vartheta$, for any solution of Problem 3, then the strategy U^* (2.1), (2.3) constructed for the arbitrary solution $u^*(\cdot), v^*(\cdot)$, will be a solution of Problem 2. Here Problem 1 has no solution in the class of strategies considered. However, U^* can be used as the basis for constructing a minimizing sequence of strategies.

Next we shall use the strategies U^* and V^* (2.1)–(2.3) to explain the meaning of the functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ introduced in the beginning of Sect.1 while determining the strategies. As we have already said, the strategy U^* is constructed in such a manner that, after disclosing it the first player attempts to induce the second player to trace the trajectory $x^*(\cdot)$. Here the Euler polygonal lines, to whose formation both players contribute, must not emerge outside the limits of the ε-tubes constructed along the corresponding trajectories. The disclosure by the first player of a suitably chosen function $\beta_1(\cdot)$ enables the second player to choose, in his turn, the function $\beta_2(\cdot)$, so that when Δ_1 and Δ_2 are partitioned into steps not exceeding $\beta_1(\varepsilon_1)$ and $\beta_2(\varepsilon_2)$ respectively, where ε_1 and ε_2 are the chosen values of the accuracy parameters of the players $(\varepsilon_2 \leqslant \varepsilon_1)$, the Euler polygonal lines will not emerge outside the limits of the tubes mentioned above.

Notes. 1°. The concept of using a penalty strategy in constructing the optimal strategy of the upper level player was proposed in /5/ for static hierarchical games, and developed further in /6/ for the dynamic hierarchical games.

2°. In constructing the strategy U^*(2.1) it is essential that the penalty strategy specified by the function $u^{(2)*}(t, x, \varepsilon)$, is universal /2/, i.e. suitable for all positions which may be encountered during the game.

3°. We can waive the assumption that the vectogram $Q^*(t, x)$(1.9) is convex. Then a solution of Problem 3 exists in the class of functions with values from the set of probabilistic measures on $P \times Q$.

3. The strategies U^*, V^* (2.1)–(2.3) form, using the terminology of /9, 10/, a conditional equilibrium strategy of two players. It can be either a coalitional mixed equilibrium strategy /9/, or a coalitional equilibrium counterstrategy /10/, depending on the assumptions made concerning the amount of information received by the players about the controls realized by the partner, and on whether the solution of Problem 3 is attained in the class of measurable functions or in the class of function-measures.

It is ture that the coalitional equilibrium strategies in /9, 10/ have a more complex structure resulting from the fact that the number of players is greater than two, and the penalty imposed on the deviant player by the remaining players depends on the index of the violator communicated from the outside. In the present game of two players the penalty of the second player is included in the construction of the strategy of the first player U^*, and the first player naturally makes no gain when deviating from the strategy U^*.

Let us find, amongst the solutions of Problem 3, a solution furnishing the index $\sigma_2(x(\vartheta))$, with a minimum, and denote it by $u^p(\cdot), v^p(\cdot)$. Such a solution exists under the assumptions made here. We denote by U^p, V^p the strategies of the first and second player obtained, respectively, from U^* and V^*(2.1)–(2.3) by replacing $u^*(\cdot)$ by $u^p(\cdot)$ and $v^*(\cdot)$ by $v^p(\cdot)$. It can be confirmed that the strategies U^p, V^p form a Pareto point of the set of the coalitional equilibrium strategies which cannot be improved by the first player.

4. Let the motion of a controlled system be described by an equation of the form

$$\frac{d^2\xi}{dt^2} = L(\varphi) F_1 + F_2 \tag{4.1}$$

$$\xi = \begin{Vmatrix} \xi_1 \\ \xi_2 \end{Vmatrix}, \quad L(\varphi) = \begin{Vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{Vmatrix}, \quad F_i = \begin{Vmatrix} F_{i1} \\ F_{i2} \end{Vmatrix}$$

The first player chooses the vector F_1, and the second player chooses the vector F_2 and a scalar quantity φ, i.e.

$$u = F_1, \quad v = \begin{Vmatrix} F_2 \\ \varphi \end{Vmatrix}$$

The equations are restricted by the constraints

$$\| F_i \|^2 = F_{i1}^2 + F_{i2}^2 \leqslant 1, \ |\varphi| \leqslant \varphi_0, \ 0 < \varphi_0 < \pi/2 \tag{4.2}$$

The initial conditions $\xi[t_*] = \xi_*$, $\xi^\cdot[t_*] = \xi_*^\cdot$ and the instant of game termination ϑ are given. The first(second) player tends to minimize the index $\sigma_1(\xi[\vartheta])$ $(\sigma_2(\xi[\vartheta]))$ of the form

$$\sigma_i(\xi[\vartheta]) = \| \xi[\vartheta] - a^{(i)} \|, \ i = 1, 2 \tag{4.3}$$

where $a^{(i)}$ are given points in the plane (ξ_1, ξ_2).

Equation (4.1) can be regarded as an equation of motion of a material point of unit mass in the plane (ξ_1, ξ_2), under the action of a force generated by two players. It can be confirmed that the saddle point condition in a small game does not hold for system (4.1). For this reason the classes of the player strategies are actually determined by the assumptions made about the information made available to the players concerning the controls of the partner realized at the particular instant. We shall assume that the second player knows at the instant t not only the position realised, but also the control of the first player, and can therefore formulate his control in the class of the counterstrategies /1, 2/. The first player knows only the position realised, and forms his control in the class of pure strategies. As we noted at the beginning of Sect.1, the results of the present paper remain valid in this case also; we must only replace the pure strategy of the second player by a counterstrategy. Thus at the instant t the first player chooses the control-force $F_1(t, \xi[t], \xi^\cdot[t], \varepsilon)$, and the second player and control-force $F_2(t, \xi[t], \xi^\cdot[t], F_1[t], \varepsilon)$ and control angle $\varphi(t, \xi[t], \xi^\cdot[t], F_1[t], \varepsilon)$, by which the force F_1 is rotated (we shall assume that anticlockwise rotation corresponds to a positive angle). The aim of the i-th player is to bring the material point as close as possible to the point $a^{(i)}$.

Putting in system (4.1) $y_1 = \xi_1$, $y_2 = \xi_2$, $y_3 = \xi_1^\cdot$, $y_4 = \xi_2^\cdot$ and making the change of variable $x_1 = y_1 + (\vartheta - t) y_3$, $x_2 = y_2 + (\vartheta - t) y_4$, $x_3 = y_3$, $x_4 = y_4$, we obtain a system whose first two equations have the form

$$z_1^{\cdot} = (\vartheta - t)\ (F_{11}\cos\varphi - F_{12}\sin\varphi + F_{21}) \qquad (4.4)$$
$$z_2^{\cdot} = (\vartheta - t)\ (F_{11}\sin\varphi + F_{12}\cos\varphi + F_{22})$$

The indices (4.3), in the variables z_1, z_2, take the form

$$\sigma_i(z[\vartheta]) = \|z[\vartheta] - a^{(i)}\|, \quad z = \begin{Vmatrix} z_1 \\ z_2 \end{Vmatrix}, \quad i = 1, 2 \qquad (4.5)$$

Since the indices (4.5) are determined by the values of the coordinates z_1 and z_2 only, and the right-hand sides of system (4.4) are independent of the remaining coordinates, we can conclude that it is sufficient to study the differential game in question for the truncated system (4.4) with indices (4.5). The initial conditions for system (4.4) will in this case be $z_1[t_*] = z_{*1} = \xi_{*1} - (\vartheta - t_*)\ \xi_{*1}^{\cdot}$, $z_2[t_*] = z_{*2} = \xi_{*2} - (\vartheta - t_*)\ \xi_{*2}^{\cdot}$.

Conditions A and B are assumed to hold.

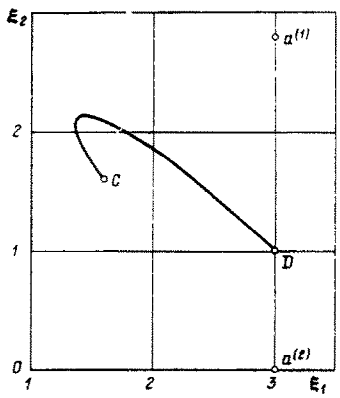
The cost function $\gamma_2(t, z)$ of the antagonistic game Γ_2 in which the actions of the players are formalized in the classes (pure strategies of the first player - counterstrategies of the second player), and the dynamics is described by system (4.4) and the functions $u^{(2)*}(t, z, \varepsilon) = F_1^{(2)*}(t, z, \varepsilon)$ and $v^{(2)*}(t, z, u, \varepsilon) = \{F_2^{(2)*}(t, z, F_1, \varepsilon), \varphi^{(2)*}(t, z, F_1, \varepsilon)$ which are analogoues of the functions (1.4), will be as follows:

$$\gamma_2(t, z) = \max\left\{\|z - a^{(2)}\| - \frac{1}{2}(1 - \cos\varphi_0)(\vartheta - t)^2, 0\right\} \qquad (4.6)$$

$$F_1^{(2)\circ} = \frac{z - a^{(2)}}{\|z - a^{(2)}\|}, \quad F_2^{(2)\circ} = -\frac{z - a^{(2)}}{\|z - a^{(2)}\|} \qquad (4.7)$$

$$\varphi^{(2)\circ} = \begin{cases} \psi, & 0 \leqslant \psi \leqslant \varphi_0 \\ -\varphi_0, & \varphi_0 \leqslant \psi \leqslant \pi \\ \varphi_0, & \pi \leqslant \psi \leqslant 2\pi - \varphi_0 \\ 2\pi - \psi, & 2\pi - \varphi_0 \leqslant \psi < 2\pi \end{cases}$$

where $\psi \in [0, 2\pi)$ is the angle between the vectors $a^{(2)} - z$ and F_1 counted from the vector $a^{(2)} - z$ in the positive direction.



Let us specify the intial conditions $t_* = 0$, $\xi_{*1} = 1{,}6$, $\xi_{*1}^{\cdot} = -0{,}4$, $\xi_{*2} = 1{,}6$, $\xi_{*2}^{\cdot} = 0{,}6$ and the following values of the parameters $\vartheta = 4$, $a_1^{(1)} = 3$, $a_2^{(1)} = 2{,}8$, $a_1^{(2)} = 3$, $a_2^{(2)} = 0$. Then we have $z_{*1} = 0$, $z_{*2} = 4$. Let $\varphi_0 = \pi/3$. We shall consider the following auxiliary problem: it is required to find a measurable vector function $p(t)$, $t_* \leqslant t \leqslant \vartheta$ furnishing the quantity $\|z(\vartheta) - a^{(1)}\|$ with a minimum under the condition

$$\|z(\vartheta) - a^{(2)}\| \leqslant \gamma_2(t, z(t)), \ t \in [t_*, \vartheta]$$

where $z(\cdot)$ is the trajectory of the system

$$z^{\cdot}(t) = p(t), \|p(t)\| \leqslant 2(\vartheta - t) \qquad (4.8)$$

satisfying the initial condition $z[t_*] = z_*$.

Under the given initial conditions and values of the parameters, one of the solutions $p^*(t)$, $0 \leqslant t \leqslant 4$ of the auxiliary problem formulated above, generating the trajectory

$$z_1^*(t) = 3 - \frac{3}{16}(4 - t)^2 \qquad (4.9)$$
$$z_2^*(t) = \left(\frac{7}{256}(4 - t)^4 + \frac{1}{2}(4 - t)^2 + 1\right)^{1/2}$$

is determined in terms of the continuous functions

$$p_1^*(t) = \frac{dz_1^*(t)}{dt}, \quad p_2^*(t) = \frac{dz_2^*(t)}{dt} \qquad (4.10)$$

We note that the function (4.6) remains constant along the trajectory $z^*(\cdot)$ (4.9).

We can confirm that any pair of continuous functions $(u^*(\cdot), v^*(\cdot))$ satisfying the conditions

$$u^*(t) + v^*(t) = p^*(t), \ \|u^*(t)\| \leqslant (4 - t) \qquad (4.11)$$
$$\|v^*(t)\| \leqslant (4 - t), \ \forall t \in [0, 4]$$

furnishes a solution of Problem 3 for system (4.4) with indices (4.5), with the initial conditions and values of the parameters given. Conditions (4.11) are satisfied, in particular, by the functions $u^*(t) = v^*(t) = {}^1/_2\ p^*(t)$.

Let us fix a pair $(u^*(\cdot), v^*(\cdot))$ which is a solution of Problem 3. Since here we have case 2) of Sect.2, therefore, according to Theorem 2 the strategies U^*, V^* (2.1)-(2.3) constructed on the basis of the pair $(u^*(\cdot), v^*(\cdot))$ and the trajectory $z^*(\cdot)$ (4.9) are optimal in the hierarchical differential game with a benevolent second player.

The figure shows the motion $z^*[\cdot]$ (4.9) in the plane (ξ_1, ξ_2), representing a unique motion of a bundle generated by the optimal strategies U^* and V^*. The motion commences at $t_* = 0$ at the point C (1,6; 1,6) and ends at the instant $\vartheta = 4$ at the point D (3,0; 1,0). The following values for the player indices are obtained: $\sigma_1(z^*[4]) = 1{,}8$, $\sigma_2(z^*[4]) = 1{,}0$.

Without going into detail we note that for any arbitrarily small $\zeta > 0$, a strategy of the first player U_ζ, can be found which will guarantee to this player a result, in the hierarchical game without presupposing the benevolence of the second player, differing from the result given above by a quantity not exceeding ζ.

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